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Hopf hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster $\mathfrak{D}^{\perp}$-parallel structure Jacobi operator

Research Article

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Abstract: Regarding the generalized Tanaka-Webster connection, we considered a new notion of $\mathfrak{D}^{\perp}$-parallel structure Jacobi operator for a real hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ and proved that a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with generalized Tanaka-Webster $\mathfrak{D}^{\perp}$-parallel structure Jacobi operator is locally congruent to an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.

MSC: 53C40, 53C15

Keywords: Complex two-plane Grassmannian • Hopf hypersurface • Generalized Tanaka-Webster connection • Structure Jacobi operator

1. Introduction

Regarding real hypersurfaces with parallel curvature tensor, many differential geometers were studied either in complex projective spaces or in quaternionic projective spaces ([7, 11, 12]). From another perspective, it is interesting to classify real hypersurfaces in complex two-plane Grassmannians with parallel shape operator, structure Jacobi operator and Ricci tensor (See [5, 6, 13–18]).

As an ambient space, a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space is the unique compact irreducible Riemannian manifold being

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equipped with both the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ not containing J. Then, we could naturally consider two geometric conditions for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$, namely, that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M ([3]), where the *Reeb* vector field ξ is defined by $\xi = -JN$. N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ and the *almost contact 3-structure* vector fields ξ_ν are defined by $\xi_\nu = -J_\nu N$ ($\nu = 1, 2, 3$).

By using the result in Alekseevskii [1], Berndt and Suh [3] proved the following :

Theorem A.

Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When we consider the Reeb vector field ξ in the expression of the curvature tensor R for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, the structure Jacobi operator R_ξ can be defined in such as

$$R_\xi(X) = R(X, \xi)\xi,$$

for any tangent vector field X on M.

By using the structure Jacobi operator R_ξ, Jeong, Pérez and Suh considered a notion of *parallel structure Jacobi operator*, that is, $\nabla_X R_\xi = 0$ for any vector field X on M and gave a non-existence theorem (See [5]).

On the other hand, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be the *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. Using the formulas in Section 2 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

Moreover, the authors [6] considered the general notion of $\mathfrak{D}^\perp$*-parallel structure Jacobi operator* defined by $\nabla_{\xi_\nu} R_\xi = 0$, $\nu = 1, 2, 3$, which is weaker than the notion of the parallel structure Jacobi operator mentioned above. They gave a non-existence theorem as follows :

Theorem B.

There do not exist any connected Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}^\perp$-parallel structure Jacobi operator if the principal curvature α is constant along the direction of ξ.

Now, instead of Levi-Civita connection for real hypersurfaces in Kähler manifolds, we consider another new connection named *generalized Tanaka-Webster connection* (in short, the *g-Tanaka-Webster* connection) $\hat{\nabla}^{(k)}$ for a non-zero real number k (See [8]). This new connection $\hat{\nabla}^{(k)}$ can be regarded as a natural extension of Tanno's generalized Tanaka-Webster connection $\hat{\nabla}$ for contact metric manifolds. Actually, Tanno [20] introduced the *generalized Tanaka-Webster* connection $\hat{\nabla}$ for contact Riemannian manifolds by using the canonical connection on a nondegenerate, integrable CR manifold.

On the other hand, the original *Tanaka-Webster connection* ([19, 21]) was given as a unique affine connection on a non-degenerate, pseudo-Hermitian CR manifold associated with the almost contact structure. In particular, if a real hypersurface in a Kähler manifold satisfies $\phi A + A\phi = 2k\phi$ ($k \neq 0$), then the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

In [10], using this g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$, we considered the notion of *Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection*, that is, $\hat{\nabla}^{(k)}_\xi R_\xi = 0$. We gave a non-existence theorem as follows:

Theorem C.

There does not exist any Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection.

In this paper, motivated by Theorems B and C, we consider another new notion for g-Tanaka-Webster parallelism of the structure Jacobi operator on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, when the structure Jacobi operator R_ξ of M satisfies $(\hat{\nabla}^{(k)}_X R_\xi)Y = 0$ for any $X \in \mathfrak{D}^\perp$ and any tangent vector field Y in M. In this case, the structure Jacobi operator is said to be a *$\mathfrak{D}^\perp$-parallel structure Jacobi operator in the generalized Tanaka-Webster connection*. Naturally, such a notion of parallelism is a generalized condition that is weaker than usual parallelism of the structure Jacobi operator in the generalized Tanaka-Webster connection.

Main Theorem.
Let M be a connected orientable Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the structure Jacobi operator R_ξ is $\mathfrak{D}^\perp$-parallel in the generalized Tanaka-Webster connection, M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.

2. Preliminaries

Basic material about complex two-plane Grassmannians is well known (See [2–4]). This complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is a Riemannian homogeneous space, even a Riemannian symmetric space. Using Lie algebra, we normalize g so that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\tilde{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\tilde{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Furthermore, the Riemannian curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned} \tilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\Big\} + \sum_{\nu=1}^{3}\Big\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\Big\}, \end{aligned} \tag{2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

Now, let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M is also denoted by g and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N. Let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{4}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned} & \phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0, \\ & \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ & \phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ & \phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1} \end{aligned} \tag{5}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$, the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{aligned} &\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\ &\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi. \end{aligned} \tag{6}$$

On the other hand, from the Kähler structure J, that is, $\tilde{\nabla} J = 0$ and the quaternionic Kähler structure J_ν (see (1)), together with Gauss and Weingarten formulas it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{7}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{8}$$

$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \tag{9}$$

Using the above expression for the curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\Big\} \\ & + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu\phi Y, Z)\phi_\nu\phi X - g(\phi_\nu\phi X, Z)\phi_\nu\phi Y\Big\} \\ & - \sum_{\nu=1}^{3}\Big\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\Big\} \\ & - \sum_{\nu=1}^{3}\Big\{\eta(X)g(\phi_\nu\phi Y, Z) - \eta(Y)g(\phi_\nu\phi X, Z)\Big\}\xi_\nu + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{10}$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ and

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ & + \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\Big\} \\ & + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\Big\} + \sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\Big\}\xi_\nu. \end{aligned} \tag{11}$$

Now, let us introduce the notion of g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ on real hypersurfaces in Kähler manifolds (See [8]).

As stated in the introduction, the Tanaka-Webster connection is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR manifold (See [19, 21]). For contact metric manifolds, their associated CR structures are pseudo-Hermitian and strongly pseudo-convex, but they are not integrable in general. In this situation, Tanno [20] defined a new connection $\hat{\nabla}$ given by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X\eta)(Y)\xi - \eta(Y)\nabla_X\xi - \eta(X)\phi Y \tag{12}$$

for contact metric manifolds as a generalization of the original Tanaka-Webster connection. From such a point of view, we called this new connection $\hat{\nabla}$ the g-Tanaka-Webster one. From this, we know that the g-Tanka-Webster connection

$\hat{\nabla}$ coincides with the Tanaka-Webster connection if the associated CR structure is integrable. Moreover, since a real hypersurface M of a Kähler manifold satisfies $A\phi + \phi A = 2\phi$ if and only if M is contact metric, we have another g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for M as an extension of Tanno's connection $\hat{\nabla}$. Actually, by substituting (7) into (12), the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for M is defined by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{13}$$

for a non-zero real number k (See [8]) (Note that $\hat{\nabla}^{(k)}$ is invariant under the choice of the orientation. Namely, we may take $-k$ instead of k in (13) for the opposite orientation $-N$).

3. Key Lemma

Let us denote by $R(X, Y)Z$ the curvature tensor of M in $G_2(\mathbb{C}^{m+2})$. Then the structure Jacobi operator R_ξ of M in $G_2(\mathbb{C}^{m+2})$ can be defined by $R_\xi X = R(X, \xi)\xi$ for any vector field $X \in T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

In [5] and [6], by using the structure Jacobi operator R_ξ, the authors obtained

$$\begin{aligned}(\nabla_X R_\xi)Y &= -g(\phi AX, Y)\xi - \eta(Y)\phi AX \\ &\quad - \sum_{\nu=1}^{3}\Big[g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\ &\quad + 3\Big\{g(\phi_\nu AX, \phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi + \eta_\nu(\phi Y)\Big(\phi_\nu\phi AX - \alpha\eta(X)\xi_\nu\Big)\Big\} \\ &\quad + 4\eta_\nu(\xi)\Big\{\eta_\nu(\phi Y)AX - g(AX, Y)\phi_\nu\xi\Big\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\Big] \\ &\quad + \eta((\nabla_X A)\xi)AY + \alpha(\nabla_X A)Y - \eta((\nabla_X A)Y)A\xi - g(AY, \phi AX)A\xi - \eta(AY)(\nabla_X A)\xi - \eta(AY)A\phi AX.\end{aligned} \tag{14}$$

From this, by using (13), together with the fact that M is Hopf, it becomes

$$\begin{aligned}(\hat{\nabla}^{(k)}_X R_\xi)Y = -\sum_{\nu=1}^{3}\Big[&g(\phi_\nu AX, Y)\xi_\nu - \eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\ &+ 3\Big\{g(\phi_\nu AX, \phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi + \eta_\nu(\phi Y)\Big(\phi_\nu\phi AX - \alpha\eta(X)\xi_\nu\Big)\Big\} \\ &+ 4\eta_\nu(\xi)\Big\{\eta_\nu(\phi Y)AX - g(AX, Y)\phi_\nu\xi\Big\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y \\ &+ \eta_\nu(Y)\eta_\nu(\phi AX)\xi - \eta_\nu(\xi)\eta(Y)\eta_\nu(\phi AX)\xi \\ &+ 3\eta(\phi_\nu Y)g(\phi AX, \phi_\nu\xi)\xi + \eta_\nu(\xi)g(\phi AX, \phi_\nu\phi Y)\xi \\ &- \eta_\nu(Y)\eta_\nu(\xi)\phi AX + \eta_\nu^2(\xi)\eta(Y)\phi AX - \eta_\nu(\xi)\eta(\phi_\nu\phi Y)\phi AX \\ &- k\eta(X)\eta_\nu(Y)\phi\xi_\nu - 4k\eta(X)\eta(\phi_\nu Y)\eta_\nu(\xi)\xi - 4k\eta(X)\eta(\phi_\nu Y)\xi_\nu \\ &+ 3\eta(Y)\eta(\phi_\nu\phi AX)\phi_\nu\xi - \eta(Y)\eta_\nu(\xi)\phi_\nu AX + \alpha\eta(X)\eta(Y)\eta_\nu(\xi)\phi_\nu\xi \\ &+ 3k\eta(X)\eta(\phi_\nu\phi Y)\phi_\nu\xi + k\eta(X)\eta(Y)\eta_\nu(\xi)\phi_\nu\xi\Big] \\ &+ \eta((\nabla_X A)\xi)AY + \alpha(\nabla_X A)Y - \alpha\eta((\nabla_X A)Y)\xi \\ &- \alpha\eta(Y)(\nabla_X A)\xi - \alpha k\eta(X)\phi AY + \alpha k\eta(X)A\phi Y\end{aligned} \tag{15}$$

for any tangent vector fields X and Y on M.

Let us assume that the structure Jacobi operator R_ξ of a Hopf hypersurface M in a complex two-plane Grassmann manifold $G_2(\mathbb{C}^{m+2})$ is $\mathfrak{D}^\perp$-*parallel* in the generalized Tanaka-Webster connection, that is,

$$(\hat{\nabla}^{(k)}_X R_\xi)Y = 0 \tag{$*$}$$

for any $X \in \mathfrak{D}^\perp$ and any tangent vector field Y on M.

Before getting our result, it is an important step to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ such that $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp$ in $G_2(\mathbb{C}^{m+2})$ when the structure Jacobi operator is $\mathfrak{D}^\perp$-parallel in the generalized Tanaka-Webster connection.

From now on, unless otherwise stated, we may put the Reeb vector field ξ as follows:

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{$**$}$$

for some unit vector fields $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$.

Now using the condition ($*$) and ($**$), we prove the following:

Lemma 3.1.
Let M be a Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}^\perp$-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. Then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

Proof. By taking the inner product with ξ in (15), it becomes

$$8k\eta(X)\eta(\phi_1 Y)\eta_1(\xi) = 0$$

for any $X \in \mathfrak{D}^\perp$ and any tangent vector field Y on M.

Thus putting $X = \xi_1 \in \mathfrak{D}^\perp$ and substituting Y with $\phi_1\xi$, it follows

$$-8k\eta^2(\xi_1)\eta^2(X_0) = 0.$$

Since k is a nonzero real number, we get $\eta(X_0) = 0$ or $\eta_1(\xi) = 0$. It means that ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$. Consequently, this completes the proof of our Lemma. □

4. Proof of Main Theorem

Let us consider a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}^\perp$-parallel structure Jacobi operator R_ξ in the generalized Tanaka-Webster connection, that is, $(\hat{\nabla}^{(k)}_X R_\xi)Y = 0$ for any $X \in \mathfrak{D}^\perp$ and any tangent vector field Y on M. Then by Lemma 3.1 we shall divide our consideration in two cases depending on whether the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$ or the distribution $\mathfrak{D}$.

First of all, we consider the case $\xi \in \mathfrak{D}^\perp$. Without loss of generality, we may put $\xi = \xi_1$. Using this notion of $\mathfrak{D}^\perp$-parallel structure Jacobi operator in the generalized Tanaka-Webster connection, we get the following:

Lemma 4.1.
If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then there does not exist any Hopf hypersurface M in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}^\perp$-parallel structure Jacobi operator in the generalized Tanaka-Webster connection.

Proof. Since by assumption ξ belongs to the distribution $\mathfrak{D}^{\perp}$, putting $X = \xi$ in (15) and using (6), we have

$$\begin{aligned}
0 =& -\Big\{\alpha g(\phi_2\xi, Y)\xi_2 + \alpha g(\phi_3\xi, Y)\xi_3 + \alpha\eta_2(Y)\phi_2\xi + \alpha\eta_3(Y)\phi_3\xi \\
&+ 3\alpha g(\phi_2\xi, \phi Y)\phi_2\xi + 3\alpha g(\phi_3\xi, \phi Y)\phi_3\xi - 3\alpha\eta_2(\phi Y)\xi_2 \\
&- 3\alpha\eta_3(\phi Y)\xi_3 - k\eta_2(Y)\phi\xi_2 - k\eta_3(Y)\phi\xi_3 - 4k\eta(\phi_2 Y)\xi_2 \\
&- 4k\eta(\phi_3 Y)\xi_3 + 3k\eta(\phi_2\phi Y)\phi_2\xi + 3k\eta(\phi_3\phi Y)\phi_3\xi\Big\} \\
&+ \eta((\nabla_\xi A)\xi)AY + \alpha(\nabla_\xi A)Y - \alpha\eta((\nabla_\xi A)Y)\xi \\
&- \alpha\eta(Y)(\nabla_\xi A)\xi - \alpha k\phi AY + \alpha kA\phi Y \\
=& -8k\eta_2(Y)\xi_3 + 8k\eta_3(Y)\xi_2 + \eta((\nabla_\xi A)\xi)AY + \alpha(\nabla_\xi A)Y \\
&- \alpha\eta((\nabla_\xi A)Y)\xi - \alpha\eta(Y)(\nabla_\xi A)\xi - \alpha k\phi AY + \alpha kA\phi Y
\end{aligned}$$

for any tangent vector field Y on M. Taking the inner product with X, we have

$$\begin{aligned}
0 = g((\hat{\nabla}^{(k)}_\xi R_\xi)Y, X) =& -8k\eta_2(Y)\eta_3(X) + 8k\eta_3(Y)\eta_2(X) + \eta((\nabla_\xi A)\xi)g(AY, X) + \alpha g((\nabla_\xi A)Y, X) \\
&- \alpha\eta(X)\eta((\nabla_\xi A)Y) - \alpha\eta(Y)g((\nabla_\xi A)\xi, X) - \alpha kg(\phi AY, X) + \alpha kg(A\phi Y, X)
\end{aligned} \tag{16}$$

for any tangent vector fields X and Y on M. Interchanging X with Y in the above equation, we get

$$\begin{aligned}
0 = g((\hat{\nabla}^{(k)}_\xi R_\xi)X, Y) =& -8k\eta_2(X)\eta_3(Y) + 8k\eta_3(X)\eta_2(Y) + \eta((\nabla_\xi A)\xi)g(AX, Y) + \alpha g((\nabla_\xi A)X, Y) \\
&- \alpha\eta(Y)\eta((\nabla_\xi A)X) - \alpha\eta(X)g((\nabla_\xi A)\xi, Y) - \alpha kg(\phi AX, Y) + \alpha kg(A\phi X, Y)
\end{aligned} \tag{17}$$

for any tangent vector fields X and Y on M. Thus subtracting (17) from (16), we obtain

$$0 = g((\hat{\nabla}^{(k)}_\xi R_\xi)Y, X) - g((\hat{\nabla}^{(k)}_\xi R_\xi)X, Y) = 16k\eta_2(X)\eta_3(Y) - 16k\eta_3(X)\eta_2(Y) \tag{18}$$

for any tangent vector fields X and Y on M. Since k is a nonzero real number, the equation (18) reduces to

$$\eta_2(X)\eta_3(Y) - \eta_3(X)\eta_2(Y) = 0$$

for any tangent vector fields X and Y on M. Replacing X with ξ_2 and Y with ξ_3, we have

$$\eta_2(\xi_2)\eta_3(\xi_3) = 0. \tag{19}$$

Let $\{e_1, e_2, \cdots, e_{4m-4}, e_{4m-3}, e_{4m-2}, e_{4m-1}\}$ be an orthonormal basis for a tangent vector space T_xM at any point $x \in M$. Without loss of generality, we may put $e_{4m-3} = \xi_1, e_{4m-2} = \xi_2$ and $e_{4m-1} = \xi_3$. Since the dimension of M is equal to $4m-1$, the above equation (19) gives a contradiction. So, we have proved our Lemma 4.1. □

Next we consider the other case $\xi \in \mathfrak{D}$. Using Theorem A, Lee and Suh [9] gave a characterization of real hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

Theorem D.
Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$.

From Lemma 3.1 and Theorem D, we see that M is locally congruent to a model space of type (B) in Theorem A under the assumption of our Main Theorem given in the introduction.

Hence it remains to check whether the structure Jacobi operator R_ξ of a real hypersurface of type (B) satisfies the condition ($*$) or not. In order to do this, we introduce a proposition concerning the eigenspaces of the model space of type (B) with respect to the shape operator. The following proposition [3] is well known : a real hypersurface M of type (B) has five distinct constant principal curvatures as follows,

Proposition.

Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \mathrm{Span}\{\xi\}, \\ &T_\beta = \mathfrak{J}J\xi = \mathrm{Span}\{\xi_\nu \mid \nu = 1,2,3\}, \\ &T_\gamma = \mathfrak{J}\xi = \mathrm{Span}\{\phi_\nu\xi \mid \nu = 1,2,3\}, \\ &T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^\perp$ is the orthogonal complement of $\mathbb{H}\mathbb{C}\xi$ where

$$\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

To check this problem, we suppose that M has a $\mathfrak{D}^\perp$-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. By putting $\xi \in \mathfrak{D}$ in (15), this equation becomes

$$\begin{aligned} (\hat{\nabla}^{(k)}_X R_\xi)Y = &-\sum_{\nu=1}^{3}\Big[\beta g(\phi_\nu X, Y)\xi_\nu + \beta\eta_\nu(Y)\phi_\nu X \\ &+ 3\Big\{\beta g(\phi_\nu X, \phi Y)\phi_\nu\xi + \beta\eta(Y)\eta_\nu(X)\phi_\nu\xi + \beta\eta_\nu(\phi Y)\phi_\nu\phi X\Big\} \\ &+ 3\beta\eta(\phi_\nu Y)g(\phi X, \phi_\nu\xi)\xi + 3\beta\eta(Y)\eta(\phi_\nu\phi X)\phi_\nu\xi\Big] \\ &+ \alpha(\nabla_X A)Y - \alpha\eta((\nabla_X A)Y)\xi - \alpha\eta(Y)(\nabla_X A)\xi \end{aligned} \tag{20}$$

for any $X \in \mathfrak{D}^\perp$ and any tangent vector field Y on M.

Case I : $Y = \xi \in T_\alpha$.

By putting $Y = \xi$ in (20) and using (4) and (6), we have

$$-\sum_{\nu=1}^{3}\Big\{3\beta\eta_\nu(X)\phi_\nu\xi - 3\beta\eta_\nu(X)\phi_\nu\xi\Big\} = 0.$$

Case II : $Y \in T_\beta$, where $T_\beta = \text{Span}\{\,\xi_i \mid i = 1,2,3\,\}$.

By setting $Y = \xi_i$, $i = 1,2,3$ in (20) and using (5), we know

$$\begin{aligned}
&-\sum_{\nu=1}^{3}\Big\{\beta g(\phi_\nu X,\xi_i)\xi_\nu + \beta\eta_\nu(\xi_i)\phi_\nu X\Big\} + \alpha(\nabla_X A)\xi_i - \alpha\eta((\nabla_X A)\xi_i)\xi \\
&= -\beta g(\phi_i X,\xi_i)\xi_i - \beta g(\phi_{i+1}X,\xi_i)\xi_{i+1} - \beta g(\phi_{i+2}X,\xi_i)\xi_{i+2} - \beta\phi_i X + \alpha(\nabla_X A)\xi_i - \alpha\eta((\nabla_X A)\xi_i)\xi \\
&= -\beta g(X,\xi_{i+2})\xi_{i+1} + \beta g(X,\xi_{i+1})\xi_{i+2} - \beta\phi_i X + \alpha(\nabla_X A)\xi_i - \alpha\eta((\nabla_X A)\xi_i)\xi
\end{aligned} \tag{21}$$

for any $X \in \mathfrak{D}^\perp$.

On the other hand, differentiating $A\xi_i = \beta\xi_i$ along X and using (8), we get

$$\begin{aligned}
(\nabla_X A)\xi_i &= \beta\nabla_X\xi_i - A\nabla_X\xi_i = \beta\Big\{q_{i+2}(X)\xi_{i+1} - q_{i+1}(X)\xi_{i+2} + \phi_i AX\Big\} - A\Big\{q_{i+2}(X)\xi_{i+1} - q_{i+1}(X)\xi_{i+2} + \phi_i AX\Big\} \\
&= \beta^2\phi_i X - \beta A\phi_i X = 0,
\end{aligned}$$

because $\phi_i X \in T_\beta$. Then the equation (21) is written as

$$-\beta g(X,\xi_{i+2})\xi_{i+1} + \beta g(X,\xi_{i+1})\xi_{i+2} - \beta\phi_i X. \tag{22}$$

<u>Subcase II-1</u> : $X = \xi_i$ in (22).

$$-\beta g(\xi_i,\xi_{i+2})\xi_{i+1} + \beta g(\xi_i,\xi_{i+1})\xi_{i+2} - \beta\phi_i\xi_i = 0.$$

<u>Subcase II-2</u> : $X = \xi_{i+1}$ in (22).

$$-\beta g(\xi_{i+1},\xi_{i+2})\xi_{i+1} + \beta g(\xi_{i+1},\xi_{i+1})\xi_{i+2} - \beta\phi_i\xi_{i+1} = 0,$$

because $\phi_i\xi_{i+1} = \xi_{i+2}$.

<u>Subcase II-3</u> : $X = \xi_{i+2}$ in (22).

$$-\beta g(\xi_{i+2},\xi_{i+2})\xi_{i+1} + \beta g(\xi_{i+2},\xi_{i+1})\xi_{i+2} - \beta\phi_i\xi_{i+2} = 0,$$

because $\phi_i\xi_{i+2} = -\xi_{i+1}$.

Summing up the above three subcases, we deduce that the structure Jacobi operator R_ξ of M is $\mathfrak{D}^\perp$-parallel on T_β in the generalized Tanaka-Webster connection.

Case III : $Y \in T_\gamma$, where $T_\gamma = \text{Span}\{\,\phi_i\xi \mid i = 1,2,3\,\}$.

By putting $Y = \phi_i\xi$ in (20) and using $\phi_\nu X \in T_\beta$ and (6), we have

$$\begin{aligned}
&-\sum_{\nu=1}^{3}\Big\{-3\beta g(\phi_\nu X,\xi_i)\phi_\nu\xi + 3\beta\eta_\nu(\phi\phi_i\xi)\phi_\nu\phi X + 3\beta\eta(\phi_\nu\phi_i\xi)g(\phi X,\phi_\nu\xi)\xi\Big\} + \alpha(\nabla_X A)\phi_i\xi - \alpha\eta((\nabla_X A)\phi_i\xi)\xi \\
&= 3\beta g(X,\xi_{i+2})\phi_{i+1}\xi - 3\beta g(X,\xi_{i+1})\phi_{i+2}\xi + 3\beta\phi_i\phi X + 3\beta g(X,\xi_i)\xi + \alpha(\nabla_X A)\phi_i\xi - \alpha\eta((\nabla_X A)\phi_i\xi)\xi
\end{aligned} \tag{23}$$

for any $X \in \mathfrak{D}^\perp$.

On the other hand, differentiating $A\phi_i\xi = \gamma\phi_i\xi$ along X and using (9), we get

$$(\nabla_X A)\phi_i\xi = -\beta A\phi_i\phi X.$$

Therefore, the equation (23) can be written as

$$3\beta g(X,\xi_{i+2})\phi_{i+1}\xi - 3\beta g(X,\xi_{i+1})\phi_{i+2}\xi + 3\beta\phi_i\phi X + 3\beta g(X,\xi_i)\xi - \alpha\beta A\phi_i\phi X - \alpha^2\beta g(X,\xi_i)\xi. \tag{24}$$

By using (5) and (6), we check easily the following subcases.

Subcase III-1 : $X = \xi_i$ in (24).

$$3\beta g(\xi_i,\xi_{i+2})\phi_{i+1}\xi - 3\beta g(\xi_i,\xi_{i+1})\phi_{i+2}\xi + 3\beta\phi_i\phi\xi_i + 3\beta g(\xi_i,\xi_i)\xi - \alpha\beta A\phi_i\phi\xi_i - \alpha^2\beta g(\xi_i,\xi_i)\xi = 0.$$

Subcase III-2 : $X = \xi_{i+1}$ in (24).

$$3\beta g(\xi_{i+1},\xi_{i+2})\phi_{i+1}\xi - 3\beta g(\xi_{i+1},\xi_{i+1})\phi_{i+2}\xi + 3\beta\phi_i\phi\xi_{i+1} + 3\beta g(\xi_{i+1},\xi_i)\xi - \alpha\beta A\phi_i\phi\xi_{i+1} - \alpha^2\beta g(\xi_{i+1},\xi_i)\xi = 0.$$

Subcase III-3 : $X = \xi_{i+2}$ in (24).

$$3\beta g(\xi_{i+2},\xi_{i+2})\phi_{i+1}\xi - 3\beta g(\xi_{i+2},\xi_{i+1})\phi_{i+2}\xi + 3\beta\phi_i\phi\xi_{i+2} + 3\beta g(\xi_{i+2},\xi_i)\xi - \alpha\beta A\phi_i\phi\xi_{i+2} - \alpha^2\beta g(\xi_{i+2},\xi_i)\xi = 0.$$

From above three subcases, we note that the structure Jacobi operator R_ξ of M is $\mathfrak{D}^\perp$-parallel on T_γ in the generalized Tanaka-Webster connection.

Case IV : $Y \in T_\lambda \oplus T_\mu$.

By putting $Y \in T_\lambda \oplus T_\mu$ in (20), we have

$$\alpha(\nabla_X A)Y - \alpha\eta((\nabla_X A)Y)\xi \tag{25}$$

for any $X = \xi_i \in \mathfrak{D}^\perp$.

On the other hand, using the Codazzi equation (11), we obtain

$$(\nabla_{\xi_i} A)Y = (\nabla_Y A)\xi_i + \sum_{\nu=1}^{3} \eta_\nu(\xi_i)\phi_\nu Y.$$

And by differentiating $A\xi_i = \beta\xi_i$ along Y and using (8), we get

$$(\nabla_Y A)\xi_i = \beta\nabla_Y\xi_i - A\nabla_Y\xi_i = \beta\phi_i AY - A\phi_i AY.$$

Since the structure Jacobi operator must be g-Tanaka-Webster $\mathfrak{D}^\perp$-parallel, the equation (25) is written as

$$\alpha\beta\phi_i AY - \alpha A\phi_i AY + \alpha\phi_i Y - \alpha\beta\eta(\phi_i AY)\xi + \alpha\eta(A\phi_i AY)\xi = 0. \tag{26}$$

Subcase IV-1 : $Y \in T_\lambda$.

By setting $Y \in T_\lambda$ in (26), we get

$$\alpha\beta\lambda\phi_i Y - \alpha\lambda^2\phi_i Y + \alpha\phi_i Y - \alpha\beta\lambda\eta(\phi_i Y)\xi + \alpha^2\lambda\eta(\phi_i Y)\xi = 0,$$

because $\phi_i Y \in T_\lambda$.

By taking the inner product with $\phi_i Y$ and using principal curvatures in the above proposition, we obtain

$$\alpha(\beta\lambda - \lambda^2 + 1) = 0.$$

Subcase IV-2: $Y \in T_\mu$.

By setting $Y \in T_\mu$ in (26), we know

$$\alpha\beta\mu\phi_i Y - \alpha\mu^2\phi_i Y + \alpha\phi_i Y - \alpha\beta\mu\eta(\phi_i Y)\xi + \alpha^2\mu\eta(\phi_i Y)\xi = 0.$$

Similarly, we have

$$\alpha(\beta\mu - \mu^2 + 1) = 0.$$

From the above two subcases, we note that the structure Jacobi operator R_ξ of M is $\mathfrak{D}^\perp$-parallel on $T_\lambda \oplus T_\mu$ in the generalized Tanaka-Webster connection.

Hence summing up these assertions, we have given a complete proof of our main theorem in the introduction.

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